

CENTRAL LIMIT THEOREM IN HÖLDER SPACES IN THE TERMS OF MAJORIZING MEASURES.

E.Ostrovsky^a, L.Sirota^b

^a Corresponding Author. Department of Mathematics and computer science,
Bar-Ilan University, 84105, Ramat Gan, Israel.
E-mail: eugostrovsky@list.ru

^b Department of Mathematics and computer science. Bar-Ilan University, 84105,
Ramat Gan, Israel.
E-mail: sirota3@bezeqint.net

ABSTRACT.

We obtain some sufficient conditions for the Central Limit Theorem for the random processes (fields) with values in the separable part of Hölder space in the modern terms of majorizing (minorizing) measures, belonging to X.Fernique and M.Talagrand.

We introduce a new class of Banach spaces-rectangle Hölder spaces and investigate CLT in this spaces via the fractional order Sobolev-Grand Lebesgue norms.

Our further considerations based on the improvement of the L.Arnold and P.Imkeller generalization of the classical Garsia-Rodemich-Rumsey inequality, which allow us to reduce degree of the distance in the important particular cases.

Key words and phrases: Majorizing and minorizing measures, Central Limit Theorem (CLT) in Banach space, upper and lower estimates, module of continuity, natural function, Hölder space, embedding, moments, natural distance, ball, rectangle difference, spaces and rectangle distance, Arnold-Imkeller and Garsia-Rodemich-Rumsey inequalities, fundamental function, covariation function, Bilateral Grand Lebesgue spaces.

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1 Notations. Hölder spaces. Statement of problem. History.

Let $(X = \{x\}, d)$ be compact metric space relative some distance (or semi - distance) $d = d(x_1, x_2)$. The Hölder (Lipshitz) space $H^o(d)$ consists by definition on all the numerical (real or complex) continuous relative the distance $d = d(t, s)$ functions $f : T \rightarrow R$ satisfying the condition

$$\lim_{\delta \rightarrow 0+} \frac{\omega(f, d, \delta)}{\delta} = 0. \quad (1.1)$$

Here $\omega(f, \delta)$ is uniform module of continuity of the (continuous) function f :

$$\omega(f, d, \delta) = \omega(f, \delta) = \sup_{t, s: d(t, s) \leq \delta} |f(t) - f(s)|. \quad (1.2)$$

The norm of the space $H^o(\omega)$ is defined as follows:

$$\|f\|_{H^o(d)} = \sup_{t \in T} |f(t)| + \sup_{d(x_1, x_2) > 0} \left\{ \frac{|f(x_1) - f(x_2)|}{d(x_1, x_2)} \right\}. \quad (1.3)$$

The detail investigation of these spaces with applications in the theory of non - linear singular integral equations is undergoing in the first chapter of a monograph of Gusejnov A.I., Muchtarov Ch.Sh. [49]. We itemize some used facts about these spaces.

This modification of the classical Hölder (Lipshitz) space is Banach space, i.e. is linear, normed, complete and separable.

Note but the space $H^o(d)$ may be trivial, i.e. may consists only constant functions. Let for instance, X be convex connected closed bounded domain in the space R^m , $m = 1, 2, \dots$ and let $d(x_1, x_2)$ be usual Euclidean distance. Then the space $H^o(d)$ is trivial: $\dim H^o(d) = 1$.

The space $H^o(d^\beta)$, $\beta = \text{const} \in (0, 1)$ in this example in contradiction is not trivial.

Further, if an another distance $r = r(x_1, x_2)$ on the source set X is such that

$$\forall x_1 \in X \Rightarrow \lim_{d(x, x_1) \rightarrow 0} \frac{d(x, x_1)}{r(x, x_1)} = 0, \quad (1.4)$$

then the space $H^o(d)$ is continuously embedded in the space $H^o(r)$.

We will write the equality (1.4) as follows: $d \ll r$.

For instance, the distance $r(x_1, x_2)$ may has a form

$$r(x_1, x_2) = d^\beta(x_1, x_2), \quad \beta = \text{const} \in (0, 1).$$

Let $\xi = \xi(x)$, $x \in X$ be *in the sequel, during whole article* be separable numerical centered (mean zero) random process (r.pr) or equally random field (r.f.) with finite (bounded) covariation function

$$R(x_1, x_2) = R_\xi(x_1, x_2) = \text{cov}(\xi(x_1), \xi(x_2)) = \mathbf{E}\xi(x_1) \cdot \xi(x_2). \quad (1.5)$$

Let also $\xi_i(x)$, $i = 1, 2, 3, \dots$ be independent copies of r.f. $\xi(x)$, defined may be on some sufficiently rich probability space,

$$S_n(x) = n^{-1/2} \sum_{i=1}^n \xi_i(x).$$

Evidently, the finite - dimensional distributions of the sequence of the r.f. $S_n(x)$ converge as $n \rightarrow \infty$ to the finite - dimensional distribution of the Gaussian separable mean zero r.f. $S_\infty(x)$ with at the same covariation function $R_\xi(x_1, x_2)$.

Definition 1.1. The r.f. $\xi(x)$ or equally the sequence of normed r.f. $\{\xi_i(x)\}$ satisfies by definition the Central Limit Theorem (CLT) in the space $H^o(d)$ (or analogously in arbitrary another separable Banach space) iff

1. $\mathbf{P}(\xi(\cdot) \in H^o(d)) = 1$;
2. The limiting Gaussian r.f. $S_\infty(x)$ belongs also to this space $H^o(d)$ a.e.
3. The sequence of distributions of the r.f. $S_n(\cdot)$ in the space $H^o(d)$ converges weakly as $n \rightarrow \infty$ to the distribution of the r.f. $S_\infty(x)$.

The last statement denotes that for arbitrary continuous bounded functional $F : H^o(d) \rightarrow R$

$$\lim_{n \rightarrow \infty} \mathbf{E}F(S_n(\cdot)) = \mathbf{E}F(S_\infty).$$

In particular,

$$\lim_{n \rightarrow \infty} \mathbf{P}(\|S_n(\cdot)\|_{H^o(d)} > u) = \mathbf{P}(\|S_\infty\|_{H^o(d)} > u), \quad u > 0.$$

Our aim in this article is obtaining some sufficient condition for CLT in Hölder space in the too modern terms of majorizing (minorizing) measures.

There are many works containing the CLT in Banach spaces, see e.g. monographs [9], [25], [28]. The recent version for CLT in Hölder spaces, for example for the Banach space valued random processes, formulated in fact in the entropy terms see in [28], chapter 4, section 4.13. (1999); [51]-[55] (2004-2006); [50], (2007).

In the article [52] is obtained the necessary and sufficient condition in entropy terms for the Hölderian functional central limit theorem.

A very important applications of this CLT in the epidemic change statistics is described in [53], [54]. Another possible applications for the functional CLT appears in the parametric Monte-Carlo method, [13], [16], [36].

In the article of B.Heinkel [17] is obtained sufficient condition for CLT in the space of continuous functions $C(T, d)$ in the more modern and more strong terms of "majorizing measures" or equally "generic changing"; see [9], [10], [44]- [47].

Notice that the CLT in the space $C(T, d)$ follows the CLT in *some* Hölder space $H^o(r)$, $d < r$. [28], chapter 4, section 4.13.

It is interest by our opinion to obtain the conditions for CLT also in the Hölder spaces in these terms.

2 Majorizing and minorizing measures.

We recall here for reader convenience some used further facts about the theory of majorizing and minorizing measures. This classical definition with theory explanation and applications basically in the investigation of local structure of random processes and fields belongs to X.Fernique [10], [11], [12] and M.Talagrand [44], [45], [46], [47], [48]. See also [4], [5], [6], [9], [25], [30], [31], [32], [37].

Let $(X, d), (Y, \rho)$ be separable metric spaces, m be arbitrary distribution, i.e. Radon probabilistic measure on the set X , $f : X \rightarrow Y$ be (measurable) function. Let also $\Phi(z)$, $z \geq 0$ be continuous Young-Orlicz function, i.e. strictly increasing function such that

$$\Phi(z) = 0 \Leftrightarrow z = 0; \lim_{z \rightarrow \infty} \Phi(z) = \infty.$$

We denote as usually

$$\Phi^{-1}(w) = \sup\{z, z \geq 0, \Phi(z) \leq w\}, w \geq 0$$

the inverse function to the function Φ ;

$$B(d, r, x) = B(r, x) = \{x_1 : x_1 \in X, d(x_1, x) \leq r\}, x \in X, 0 \leq r \leq \text{diam}(X)$$

be the closed ball of radii r with center at the point x .

Let us introduce the Orlicz space $L(\Phi) = L(\Phi; m \times m, X \otimes X)$ on the set $X \otimes X$ equipped with the Young-Orlicz function Φ .

We assume henceforth that for all the values $x_1, x_2 \in X$, $x_1 \neq x_2$ (the case $x_1 = x_2$ is trivial) the value $\rho(f(x_1), f(x_2))$ belongs to the space $L(\Phi)$.

As a rule,

$$\rho(f(x_1), f(x_2)) = |f(x_1) - f(x_2)|.$$

Note that for the existence of such a function $\Phi(\cdot)$ is necessary and sufficient only the integrability of the distance $\rho(f(x_1), f(x_2))$ over the product measure $m \times m$:

$$\int_X \int_X \rho(f(x_1), f(x_2)) m(dx_1) m(dx_2) < \infty,$$

see [23], chapter 2, section 8.

Under this assumption the distance $d = d(x_1, x_2)$ may be constructively defined by the formula:

$$d_\Phi(x_1, x_2) := \|\rho(f(x_1), f(x_2))\|L(\Phi), \quad (2.1)$$

where $\|\cdot\|L(\Phi)$ denotes the Orlicz's norm.

Since the function $\Phi = \Phi(z)$ is presumed to be continuous and strictly increasing, it follows from the relation (1.1) that $V(d_\Phi) \leq 1$, where by definition

$$V(d) := \int_X \int_X \Phi \left[\frac{\rho(f(x_1), f(x_2))}{d(x_1, x_2)} \right] m(dx_1) m(dx_2). \quad (2.2)$$

Let us define also the following important distance function: $w(x_1, x_2) =$

$$w(x_1, x_2; V) = w(x_1, x_2; V, m) = w(x_1, x_2; V, m, \Phi) = w(x_1, x_2; V, m, \Phi, d) \stackrel{\text{def}}{=}$$

$$6 \int_0^{d(x_1, x_2)} \left\{ \Phi^{-1} \left[\frac{4V}{m^2(B(r, x_1))} \right] + \Phi^{-1} \left[\frac{4V}{m^2(B(r, x_2))} \right] \right\} dr, \quad (2.3)$$

where $m(\cdot)$ is probabilistic Borelian measure on the set X .

The triangle inequality and other properties of the distance function $w = w(x_1, x_2)$ are proved in [24].

Definition 2.1. (See [24]). The measure m is said to be *minorizing measure* relative the distance $d = d(x_1, x_2)$, if for each values $x_1, x_2 \in X$ $V(d) < \infty$ and moreover $w(x_1, x_2; V(d)) < \infty$.

We will denote the set of all minorizing measures on the metric set (X, d) by $\mathcal{M} = \mathcal{M}(\mathcal{X})$.

Evidently, if the function $w(x_1, x_2)$ is bounded, then the minorizing measure m is majorizing. Inverse proposition is not true, see [24], [2].

Remark 2.1. If the measure m is minorizing, then

$$w(x_n, x; V(d)) \rightarrow 0 \Leftrightarrow d(x_n, x) \rightarrow 0, \quad n \rightarrow \infty.$$

Therefore, the continuity of a function relative the distance d is equivalent to the continuity of this function relative the distance w .

Remark 2.2. If

$$\sup_{x_1, x_2 \in X} w(x_1, x_2; V(d)) < \infty,$$

then the measure m is called *majorizing measure*.

Some considerations about the choice of the majorizing (minorizing) measures see in the article [39]; see also reference therein.

The following important inequality belongs to L.Arnold and P.Imkeller [2], [19]; see also [21], [3].

Theorem of L.Arnold and P.Imkeller. *Let the measure m be minorizing. Then there exists a modification of the function f on the set of zero measure, which we denote also by f , for which*

$$\rho(f(x_1), f(x_2)) \leq w(x_1, x_2; V, m, \Phi, d). \quad (2.4)$$

As a consequence: this function f is d -continuous and moreover w -Lipshitz continuous with unit constant.

The inequality (2.4) of L.Arnold and P.Imkeller is significant generalization of celebrated Garsia-Rodemich-Rumsey inequality, see [15], with at the same applications as mentioned before [18], [30], [31], [32], [42].

Remark 2.3. The inequality of L.Arnold and P.Imkeller (2.4) is closely related with the theory of fractional order Sobolev's - rearrangement invariant spaces, see [3], [15], [18], [21], [27], [31], [42], [43].

Remark 2.4. In the previous articles [24], [7] was imposed on the function $\Phi(\cdot)$ the following Δ^2 condition:

$$\Phi(x)\Phi(y) \leq \Phi(K(x+y)), \quad \exists K = \text{const} \in (1, \infty), \quad x, y \geq 0$$

or equally

$$\sup_{x, y > 0} \left[\frac{\Phi^{-1}(xy)}{\Phi^{-1}(x) + \Phi^{-1}(y)} \right] < \infty. \quad (2.5)$$

We do not suppose this condition. For instance, we can consider the function of a view $\Phi(z) = |z|^p$, which does not satisfy the condition (2.5).

3 Hölder's CLT over Lebesgue-Riesz spaces.

Let $\xi = \xi(x)$, $x \in X$ be again separable centered continuous *in probability* random field (r.f), not necessary to be Gaussian. The correspondent probability and expectation we will denote by \mathbf{P} , \mathbf{E} , and the probabilistic Lebesgue-Riesz L_p norm of a random variable (r.v) η we will denote as follows:

$$|\eta|_p \stackrel{\text{def}}{=} [\mathbf{E}|\eta|^p]^{1/p}.$$

Let the r.f. $\xi(\cdot)$ be such that

$$\exists p = \text{const} \geq 2 \Rightarrow \sup_{x \in X} |\xi(x)|_p < \infty.$$

Then we can define a so-called natural, or Pisier's distance [40] $d_p = d_p(x_1, x_2)$ as follows

$$d_p(x_1, x_2) \stackrel{\text{def}}{=} |\xi(x_1) - \xi(x_2)|_p, \quad (3.0)$$

which is evidently bounded.

Theorem 3.1. Suppose the measure m and distance d_p are such that

$$m^2(B(d_p, r, x)) \geq r^\theta / C(\theta), \quad r \in [0, \text{diam}(X, d_p)], \quad \exists \theta = \text{const} > 0, \quad C(\theta) \in (0, \infty). \quad (3.1)$$

Let also $p = \text{const} > \theta$, so that $p > \max(\theta, 2)$.

Our statement: for arbitrary (semi -) distance $\rho = \rho(x_1, x_2)$ such that $d_p \ll \rho$ the r.f. $\xi(x)$ satisfies the CLT in Hölder space $H^o(\rho)$.

Proof. We will use the following proposition from the article [37] (Proposition 2.1.): we get using the inference also theorem 2.1 therein that for the r.f. $\xi = \xi(x)$ the following inequality holds: $m \in \mathcal{M}$ and

$$|\xi(x_1) - \xi(x_2)| \leq 12 Z^{1/p} 4^{1/p} C^{1/p}(\theta) \frac{d_p^{1-\theta/p}(x_1, x_2)}{1 - \theta/p}, \quad (3.2)$$

where the r.v. Z has unit expectation: $\mathbf{E}Z = 1$.

We intent to apply the inequality (3.2) for the random fields $S_n(\cdot)$ instead $\xi(x)$. Note first of all that the classical Rosenthal's inequality [56] asserts in particular that if $\{\zeta_i\}$, $i = 1, 2, \dots$ are the sequence of i., i.d. *centered* r.v. with finite p^{th} moment, then

$$\sup_n \left| n^{-1/2} \sum_{i=1}^n \zeta_i \right|_p \leq \frac{C_R p}{e \cdot \ln p} |\zeta_1|_p, \quad p \geq 2. \quad (3.3)$$

About the exact value of the constant C_R see the article [38]. Note that for the symmetrical distributed r.v. $C_R \leq 1.53573$.

We have using Rosenthal's inequality since $p \geq 2$

$$|S_n(x_1) - S_n(x_2)|_p \leq d_p(x_1, x_2) \cdot \frac{C_R p}{e \cdot \ln p} \leq C_1(p) \cdot d_p(x_1, x_2),$$

and we conclude by means of estimate (3.2)

$$|S_n(x_1) - S_n(x_2)| \leq C_2(\theta, p) \cdot Z_n \cdot \frac{d_p^{1-\theta/p}(x_1, x_2)}{1 - \theta/p} = C_3(\theta, p) \cdot Z_n^{1/p} \cdot d_p^{1-\theta/p}(x_1, x_2), \quad (3.4)$$

where Z_n is the sequence of non - negative r.v. with unit expectation $\mathbf{E}Z_n = 1$.

Let $\nu = \nu(x_1, x_2)$ be arbitrary *intermediate* distance on the set X between $r(\cdot, \cdot)$ and $d_p^{1-\theta/p}(\cdot, \cdot)$:

$$d_p^{1-\theta/p}(\cdot, \cdot) << \nu(\cdot, \cdot) << r(\cdot, \cdot).$$

We deduce from (3.4)

$$\frac{|S_n(x_1) - S_n(x_2)|}{\nu(x_1, x_2)} \leq C_3(\theta, p) \cdot Z_n^{1/p} \cdot \frac{d_p^{1-\theta/p}(x_1, x_2)}{\nu(x_1, x_2)}, \quad (3.5)$$

and we conclude taking into account the structure of compact embedded Hölder subspaces into ones that the sequence of r.f. $S_n(\cdot)$ satisfies of the famous Prokhorov's criterion [41] for the weak compactness of its distributions in the Hölder space $H^o(r)$.

This completes the proof of theorem 3.1.

4 Main result: Grand Lebesgue spaces approach.

We recall first of all briefly the definition and some simple properties of the so-called Grand Lebesgue spaces; more detail investigation of these spaces see in [14], [20], [22], [26], [28], [29]; see also reference therein.

Recently appear the so-called Grand Lebesgue Spaces $GLS = G(\psi) = G\psi = G(\psi; A, B)$, $A, B = \text{const}$, $A \geq 1$, $A < B \leq \infty$, spaces consisting on all the random variables (measurable functions) $f : \Omega \rightarrow R$ with finite norms

$$\|f\|G(\psi) \stackrel{\text{def}}{=} \sup_{p \in (A, B)} [|f|_p / \psi(p)]. \quad (4.1)$$

Here $\psi(\cdot)$ is some continuous positive on the *open* interval (A, B) function such that

$$\inf_{p \in (A, B)} \psi(p) > 0, \quad \psi(p) = \infty, \quad p \notin (A, B).$$

We will denote

$$\text{supp}(\psi) \stackrel{\text{def}}{=} (A, B) = \{p : \psi(p) < \infty, \}$$

The set of all ψ functions with support $\text{supp}(\psi) = (A, B)$ will be denoted by $\Psi(A, B)$.

This spaces are rearrangement invariant, see [8], and are used, for example, in the theory of probability [22], [28], [29]; theory of Partial Differential Equations [14], [20]; functional analysis [14], [20], [26], [29]; theory of Fourier series, theory of martingales, mathematical statistics, theory of approximation etc.

Notice that in the case when $\psi(\cdot) \in \Psi(A, \infty)$ and a function $p \rightarrow p \cdot \log \psi(p)$ is convex, then the space $G\psi$ coincides with some *exponential* Orlicz space.

Conversely, if $B < \infty$, then the space $G\psi(A, B)$ does not coincides with the classical rearrangement invariant spaces: Orlicz, Lorentz, Marcinkiewicz etc.

The fundamental function of these spaces $\phi(G(\psi), \delta) = \|I_A\|G(\psi), \text{mes}(A) = \delta, \delta > 0$, where I_A denotes as ordinary the indicator function of the measurable set A , by the formulae

$$\phi(G(\psi), \delta) = \sup_{p \in \text{supp}(\psi)} \left[\frac{\delta^{1/p}}{\psi(p)} \right]. \quad (4.2)$$

The fundamental function of arbitrary rearrangement invariant spaces plays very important role in functional analysis, theory of Fourier series and transform [8] as well as in our further narration.

Many examples of fundamental functions for some $G\psi$ spaces are calculated in [28], [29].

Remark 4.1 If we introduce the *discontinuous* function

$$\psi_{(r)}(p) = 1, \quad p = r; \quad \psi_{(r)}(p) = \infty, \quad p \neq r, \quad p, r \in (A, B)$$

and define formally $C/\infty = 0$, $C = \text{const} \in R^1$, then the norm in the space $G(\psi_r)$ coincides with the L_r norm:

$$\|f\|G(\psi_{(r)}) = |f|_r.$$

Thus, the Grand Lebesgue Spaces are direct generalization of the classical exponential Orlicz's spaces and Lebesgue spaces L_r .

Remark 4.2 The function $\psi(\cdot)$ may be generated as follows. Let $\xi = \xi(x)$ be some measurable function: $\xi : X \rightarrow R$ such that $\exists(A, B) : 1 \leq A < B \leq \infty, \forall p \in (A, B) \quad |\xi|_p < \infty$. Then we can choose

$$\psi(p) = \psi_\xi(p) = |\xi|_p.$$

Analogously let $\xi(t, \cdot) = \xi(t, x), t \in T$, T is arbitrary set, be some *family* $F = \{\xi(t, \cdot)\}$ of the measurable functions: $\forall t \in T \xi(t, \cdot) : X \rightarrow R$ such that

$$\exists(A, B) : 1 \leq A < B \leq \infty, \sup_{t \in T} |\xi(t, \cdot)|_p < \infty.$$

Then we can choose

$$\psi(p) = \psi_F(p) = \sup_{t \in T} |\xi(t, \cdot)|_p. \quad (4.3)$$

The function $\psi_F(p)$ may be called as a *natural function* for the family F . This method was used in the probability theory, more exactly, in the theory of random fields, see [22],[28], chapters 3,4.

For instance, the function $\Phi(\cdot)$ may be introduced by a natural way based on the family

$$F_{d,X} = \{d(\xi(x_1), \xi(x_2))\}, \quad x_1, x_2 \in X.$$

Remark 4.3 Note that the so-called *exponential* Orlicz spaces are particular cases of Grand Lebesgue spaces [22], [28], p. 34-37. In detail, let the N - Young-Orlicz function has a view

$$N(u) = e^{\mu(u)},$$

where the function $u \rightarrow \mu(u)$ is convex even twice differentiable function such that

$$\lim_{u \rightarrow \infty} \mu'(u) = \infty.$$

Introduce a new function

$$\psi_{\{N\}}(x) = \exp \left\{ \frac{[\log N(e^x)]^*}{x} \right\},$$

where $g^*(\cdot)$ denotes the Young-Fenchel transform of the function g :

$$g^*(x) = \sup_y (xy - g(y)).$$

Conversely, the N - function may be calculated up to equivalence through corresponding function $\psi(\cdot)$ as follows:

$$N(u) = e^{\tilde{\psi}^*(\log |u|)}, \quad |u| > 3; \quad N(u) = Cu^2, \quad |u| \leq 3; \quad \tilde{\psi}(p) = p \log \psi(p). \quad (4.4)$$

The Orlicz's space $L(N)$ over our probabilistic space is equivalent up to sublinear norms equality with Grand Lebesgue space $G\psi_{\{N\}}$.

Remark 4.4. The theory of probabilistic *exponential* Grand Lebesgue spaces or equally exponential Orlicz spaces gives a very convenient apparatus for investigation of the r.v. with exponential decreasing tails of distributions. Namely, the non-zero r.v. η belongs to the Orlicz space $L(N)$, where $N = N(u)$ is function described before, if and only if

$$\mathbf{P}(\max(\eta, -\eta) > z) \leq \exp(-\mu(Cz)), \quad z > 1, \quad C = C(N(\cdot), \|\eta\|L(N)) \in (0, \infty).$$

(Orlicz's version).

Analogously may be written a Grand Lebesgue version of this inequality. In detail, if $0 < \|\eta\|G\psi < \infty$, then

$$\mathbf{P}(\max(\eta, -\eta) > z) \leq 2 \exp\left(-\tilde{\psi}(\log[z/\|\eta\|G\psi])\right), \quad z \geq \|\eta\|G\psi.$$

Conversely, if

$$\mathbf{P}(\max(\eta, -\eta) > z) \leq 2 \exp\left(-\tilde{\psi}(\log[z/K])\right), \quad z \geq K,$$

then $\|\eta\|G\psi \leq C(\psi) \cdot K$, $C(\psi) \in (0, \infty)$.

A very important subclass of the $G\psi$ spaces form the so-called $B(\phi)$ spaces.

Let $\phi = \phi(\lambda)$, $\lambda \in (-\lambda_0, \lambda_0)$, $\lambda_0 = \text{const} \in (0, \infty]$ be some even strong convex which takes positive values for positive arguments twice continuous differentiable function, such that

$$\phi(0) = 0, \quad \phi''(0) \in (0, \infty), \quad \lim_{\lambda \rightarrow \lambda_0} \phi(\lambda)/\lambda = \infty. \quad (4.5)$$

We denote the set of all these function as Φ ; $\Phi = \{\phi(\cdot)\}$.

We say that the *centered* random variable (r.v) ξ belongs to the space $B(\phi)$, if there exists some non-negative constant $\tau \geq 0$ such that

$$\forall \lambda \in (-\lambda_0, \lambda_0) \Rightarrow \mathbf{E} \exp(\lambda \xi) \leq \exp[\phi(\lambda \tau)]. \quad (4.6)$$

The minimal value τ satisfying (4.6) is called a $B(\phi)$ norm of the variable ξ , write

$$\|\xi\|B(\phi) = \inf\{\tau, \tau > 0 : \forall \lambda \Rightarrow \mathbf{E} \exp(\lambda \xi) \leq \exp(\phi(\lambda \tau))\}.$$

This spaces are very convenient for the investigation of the r.v. having a exponential decreasing tail of distribution, for instance, for investigation of the limit theorem, the exponential bounds of distribution for sums of random variables, non-asymptotical properties, problem of continuous of random fields, study of Central Limit Theorem in the Banach space etc.

The space $B(\phi)$ with respect to the norm $\|\cdot\|B(\phi)$ and ordinary operations is a Banach space which is isomorphic to the subspace consisted on all the centered variables of Orlicz's space $(\Omega, F, \mathbf{P}), N(\cdot)$ with N - function

$$N(u) = \exp(\phi^*(u)) - 1, \quad \phi^*(u) = \sup_{\lambda} (\lambda u - \phi(\lambda)). \quad (4.7)$$

The transform $\phi \rightarrow \phi^*$ is called Young-Fenchel transform. The proof of considered assertion used the properties of saddle-point method and theorem of Fenchel-Morau:

$$\phi^{**} = \phi.$$

The next facts about the $B(\phi)$ spaces are proved in [22], [28], p. 19-40:

$$1. \xi \in B(\phi) \Leftrightarrow \mathbf{E}\xi = 0, \text{ and } \exists C = \text{const} > 0,$$

$$U(\xi, x) \leq \exp(-\phi^*(Cx)), x \geq 0,$$

where $U(\xi, x)$ denotes in this article the *tail* of distribution of the r.v. ξ :

$$U(\xi, x) = \max(\mathbf{P}(\xi > x), \mathbf{P}(\xi < -x)), x \geq 0,$$

and this estimation is asymptotically exact.

Here and further $C, C_j, C(i)$ will denote the non-essentially positive finite "constructive" constants.

The function $\phi(\cdot)$ may be "constructively" introduced by the formula

$$\phi(\lambda) = \phi_0(\lambda) \stackrel{\text{def}}{=} \log \sup_{t \in T} \mathbf{E} \exp(\lambda \xi(t)), \quad (4.8)$$

if obviously the family of the centered r.v. $\{\xi(t), t \in T\}$ satisfies the *uniform* Kramer's condition:

$$\exists \mu \in (0, \infty), \sup_{t \in T} U(\xi(t), x) \leq \exp(-\mu x), x \geq 0. \quad (4.9)$$

In this case we will call the function $\phi(\lambda) = \phi_0(\lambda)$ *natural* function.

2. We define $\psi(p) = \psi_\phi(p) := p/\phi^{-1}(p)$, $p \geq 2$. It is proved that the spaces $B(\phi)$ and $G(\psi)$ coincides: $B(\phi) = G(\psi)$ (set equality) and both the norm $\|\cdot\|_{B(\phi)}$ and $\|\cdot\|$ are equivalent: $\exists C_1 = C_1(\phi), C_2 = C_2(\phi) = \text{const} \in (0, \infty), \forall \xi \in B(\phi)$

$$\|\xi\|_{G(\psi)} \leq C_1 \|\xi\|_{B(\phi)} \leq C_2 \|\xi\|_{G(\psi)}. \quad (4.10)$$

The Gaussian (more precisely, subgaussian) case is considered in [15], [18], [42] may be obtained by choosing $\Phi(z) = \Phi_2(z) := \exp(z^2/2) - 1$ or equally $\psi(p) = \psi_2(p) = \sqrt{p}$. It may be considered easily more general example when $\Phi(z) = \Phi_Q(z) := \exp(|z|^Q/Q) - 1$, $Q = \text{const} > 0$; $\Leftrightarrow \psi(p) = \psi_Q(p) := p^{1/Q}$, $p \geq 1$.

In the last case the following implication holds:

$$\eta \in L(\Phi_Q), Q > 1 \Leftrightarrow U(\eta, x) \leq \exp(-C(\Phi, \eta) x^{Q'}),$$

where as usually $Q' = Q/(Q - 1)$.

Assume that the number θ , measure m , distance $d_{(\psi)}$, and the function $\psi = \psi(p)$ are such that $\theta > 0$, $C = C(\theta) = \text{const} \in (0, \infty)$;

$$(A, B) := \text{supp } \psi(\cdot), A > \theta, B > A;$$

$$d_{(\psi)}(x_1, x_2) := \|\xi(x_1) - \xi(x_2)\|_{G\psi};$$

$$m^2(B(d_{(\psi)}, r, x)) \geq r^\theta / C(\theta), r \in [0, \text{diam}(X, d_{(\psi)})], C(\theta) \in (0, \infty).$$

Define also a new function:

$$\psi_\theta(p) \stackrel{\text{def}}{=} \psi(p)/(1 - \theta/p), \quad p \in (A, B).$$

Theorem 4.1. The sample paths of the r.f. $\xi(x)$ belong a.e. for all the values $p \in (A, B)$ to the Hölder space $H(d_{(\psi)}^{1-\theta/p})$. Moreover,

$$\left| \sup_{d_{(\psi)}(x_1, x_2) > 0} \frac{|\xi(x_1) - \xi(x_2)|}{d_{(\psi)}^{1-\theta/p}(x_1, x_2)} \right|_p \leq C \cdot \psi_\theta(p), \quad p \in (A, B). \quad (4.11)$$

As a consequence: let the semi - distance $\rho = \rho(x_1, x_2)$ be in addition such that

$$d_{(\psi)}^{1-\theta/A}(\cdot, \cdot) < \rho(\cdot, \cdot), \quad (4.12)$$

then the r.f. $\xi(\cdot)$ belongs to the Hölder space $H^o(\rho)$ with probability one.

Proof. We start from the relation (3.2):

$$|\xi(x_1) - \xi(x_2)| \leq 12 \, Z^{1/p} \, 4^{1/p} \, C^{1/p}(\theta) \, \frac{d_p^{1-\theta/p}(x_1, x_2)}{1 - \theta/p},$$

where Z is a non - negative r.v. $Z = Z_{(p)}$ is such that $\mathbf{E}Z \leq 1$.

It follows from the direct definition of the norm in $G\psi$ spaces

$$d_p(x_1, x_2) = |\xi(x_1) - \xi(x_2)|_p \leq \psi(p) \cdot d_{(\psi)}(x_1, x_2), \quad p \in (A, B);$$

and we derive after substituting

$$|\xi(x_1) - \xi(x_2)| \leq C_1(A, B) \cdot Z^{1/p} \cdot \psi(p) \cdot \frac{d_{(\psi)}^{1-\theta/p}(x_1, x_2)}{1 - \theta/p},$$

or equally

$$\left| \frac{|\xi(x_1) - \xi(x_2)|}{d_{(\psi)}^{1-\theta/p}(x_1, x_2)} \right| \leq C_1(A, B) \cdot Z^{1/p} \cdot \frac{\psi(p)}{1 - \theta/p} = C_1(A, B) \cdot Z^{1/p} \cdot \psi_\theta(p).$$

Since the right-hand of the last inequality does not depend on the variables x_1, x_2 ,

$$\sup_{d_{(\psi)}(x_1, x_2) > 0} \left| \frac{|\xi(x_1) - \xi(x_2)|}{d_{(\psi)}^{1-\theta/p}(x_1, x_2)} \right| \leq C_1(A, B) \cdot Z^{1/p} \cdot \psi_\theta(p). \quad (4.13)$$

It remains to calculate the L_p norm on both the sides of the last inequality.

Remark 4.5. The case when $\psi(p) = \sqrt{p}$ correspondents to the Gaussian (more generally, subgaussian) random field $\xi(x)$. The case $\psi(p) = \exp(Cp)$ appears in the articles [2] and [19]. However, in both these cases the condition (2.5) is satisfied.

In the case $\psi(p) = \psi_{(r)}(p)$ we obtain the statement of theorem 3.1. as a particular case.

Theorem 4.2. Suppose that all the conditions of theorem 4.1 are satisfied. Then the r.f. $\xi(x)$ satisfies the CLT in Hölder space $H^o(\rho)$.

Proof. We apply the statement of theorem 4.1 to the r.f. $S_n(\cdot)$.
Note first of all that

$$d_{\psi_R}(x_1, x_2) := ||S_n(x_1) - S_n(x_2)||G\psi_R \leq ||\xi(x_1) - \xi(x_2)|| = d_\psi(x_1, x_2). \quad (4.14)$$

Therefore,

$$m^2(B(d_{\psi_R}, r, x) \geq r^\theta/C(\theta), \quad x \in X.$$

Denote

$$\psi_{\theta,R}(p) = \frac{\psi_R(p)}{1 - \theta/p}. \quad (4.15)$$

It is known [26] that the function $\psi_{\theta,R}(\cdot)$ belongs to the set $\Psi = \{\psi\}$ with at the same support (A, B) .

We have by virtue of proposition of theorem 4.1

$$\sup_n \left| \sup_{d_{(\psi_R)}(x_1, x_2) > 0} \frac{|S_n(x_1) - S_n(x_2)|}{d_{(\psi_R)}^{1-\theta/p}(x_1, x_2)} \right|_p \leq C \cdot \psi_{\theta,R}(p), \quad p \in (A, B).$$

All the more so

$$\sup_n \left| \sup_{d_{(\psi_R)}(x_1, x_2) > 0} \frac{|S_n(x_1) - S_n(x_2)|}{d_{(\psi_R)}^{1-\theta/A}(x_1, x_2)} \right|_p \leq C \cdot \psi_{\theta,R}(p), \quad p \in (A, B), \quad (4.16)$$

and hence

$$\sup_n \left\| \sup_{d_{(\psi_R)}(x_1, x_2) > 0} \frac{|S_n(x_1) - S_n(x_2)|}{d_{(\psi_R)}^{1-\theta/A}(x_1, x_2)} \right\| G\psi_{\theta,R} \leq C < \infty. \quad (4.17)$$

It remains to repeat the arguments using by the proof of theorem 3.1.

5 CLT in rectangle Hölder spaces via the fractional order Sobolev-Grand Lebesgue Spaces.

Let D be convex non-empty bounded closed domain with Lipschitz boundary in the whole space R^d , $d = 1, 2, \dots$, and let $f : D \rightarrow R$ be measurable function.

We assume further for simplicity that $D = [0, 1]^d$.

We denote and define $|x| = (\sum_{i=1}^d x_i^2)^{1/2}$, $\alpha = \text{const} \in (0, 1]$,

$$|f|_p = |f|_{p,D} = \left[\int_D |f(x)|^p dx \right]^{1/p}, \quad |u(\cdot, \cdot)|_p = |u(\cdot, \cdot)|_{p,D^2} =$$

$$\left[\int_D \int_D |u(x, y)|^p dx dy \right]^{1/p}, \quad p = \text{const} \geq 1,$$

$$\omega(f, \delta) = \sup\{|f(x) - f(y)| : x, y \in D, |x - y| \leq \delta\}, \quad \delta \in [0, \text{diam}(D)], \quad (5.0)$$

$$G_\alpha[f](x, y) = \frac{f(x) - f(y)}{|x - y|^\alpha}, \quad \nu(dx, dy) = \frac{dxdy}{|x - y|}, \quad (5.1)$$

$$|u(\cdot, \cdot)|_{p, \nu} = |u(\cdot, \cdot)|_{p, \nu, D^2} = \left[\int_D \int_D |u(x, y)|^p \nu(dx, dy) \right]^{1/p}, \quad (5.2)$$

$$\|f\|W(\alpha, p) = |G_\alpha[f](\cdot, \cdot)|_{p, \nu, D^2}. \quad (5.3)$$

The norm $\|\cdot\|W(\alpha, p)$, more precisely, semi-norm is said to be *fractional* Sobolev's norm or similar *Aronszajn, Gagliardo or Slobodeckij* norm; see, e.g. [27].

If in the definition (5.3) instead the $L_p(D^2)$ stands another norm $\|\cdot\|V(D^2)$, for instance, Lorentz, Marcinkiewicz or Grand Lebesgue, (we recall its definition further), we obtain correspondingly the definition of the fractional $\|\cdot\|V(D^2)$ norm.

The inequality

$$|f(t) - f(s)| \leq 8 \cdot 4^{1/p} \cdot \left[\frac{\alpha + 1/p}{\alpha - 1/p} \right] \cdot |t - s|^{\alpha-1/p} \cdot \|f\|W(\alpha, p), \quad (5.4)$$

or equally

$$\omega(f, \delta) \leq 8 \cdot 4^{1/p} \cdot \left[\frac{\alpha + 1/p}{\alpha - 1/p} \right] \cdot \delta^{\alpha-1/p} \cdot \left[\int_D \int_D \frac{|f(x) - f(y)|^p dx dy}{|x - y|^{\alpha p + 1}} \right]^{1/p}, \quad (5.5)$$

which is true in the case $d = 1$ (the multidimensional case will be consider further), $p > 1/\alpha$, is called *fractional* Sobolev, or Aronszajn, Gagliardo, Slobodeckij inequality.

More precisely, the inequality (5.4) implies that the function f may be redefined on the set of measure zero as a continuous function for which (5.4) there holds.

Another look on the inequality (5.4): it may be construed as an imbedding theorem from the Sobolev fractional space into the space of (uniform) continuous functions on the set D .

The proof of the our version of inequality (5.4) may be obtained immediately from an article [18], which based in turn on the famous Garsia-Rodemich-Rumsey inequality, see [15].

There are many generalizations of fractional Sobolev's imbedding theorem: on the Sobolev-Orlicz's spaces [1], p. 253-364, on the so-called *integer* Sobolev-Grand Lebesgue spaces [30], on the Lorentz and Marcinkiewicz spaces etc.

The applications of these inequalities in the theory of random processes is investigated in the article [31].

The predicate that $x \in D$ imply $x = \vec{x} = (x_1, x_2, \dots, x_d)$, $0 \leq x_i \leq 1$.

We define as in [42], [18] the *rectangle difference* operator $\square[f](\vec{x}, \vec{y}) = \square[f](x, y)$, $x, y \in D$, $f : D \rightarrow R$ as follows.

$$\Delta^{(i)}[f](x, y) := f(x_1, x_2, \dots, x_{i-1}, y_i, x_{i+1}, \dots, x_d) - f(x_1, x_2, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_d),$$

with obvious modification when $i = 1$ or $i = d$;

$$\square[f](x, y) \stackrel{def}{=} \left\{ \otimes_{i=1}^d \Delta^{(i)} \right\} [f](x, y). \quad (5.6)$$

For instance, if $d = 2$, then

$$\square[f](x, y) = f(y_1, y_2) - f(x_1, y_2) - f(y_1, x_2) + f(x_1, x_2).$$

If the function $f : [0, 1]^d \rightarrow R$ is d times continuous differentiable, then

$$\square[f](\vec{x}, \vec{y}) = \int_{x_1}^{y_1} \int_{x_2}^{y_2} \cdots \int_{x_d}^{y_d} \frac{\partial^d f}{\partial x_1 \partial x_2 \cdots \partial x_d} dx_1 dx_2 \cdots dx_d.$$

The *rectangle module of continuity* $\Omega(f, \vec{\delta}) = \Omega(f, \delta)$ for the (continuous a.e.) function f and vector $\vec{\delta} = \delta = (\delta_1, \delta_2, \dots, \delta_d) \in [0, 1]^d$ may be defined as well as ordinary module of continuity $\omega(f, \delta)$ as follows:

$$\Omega(f, \vec{\delta}) \stackrel{def}{=} \sup \{ |\square[f](x, y)|, (x, y) : |x_i - y_i| \leq \delta_i, i = 1, 2, \dots, d \}.$$

Let $\vec{\alpha} = \{\alpha_k\}$, $\alpha_k \in (0, 1]$, $k = 1, 2, \dots, d$; $p > p_0 \stackrel{def}{=} \max_k (1/\alpha_k)$, $M = \text{card}\{i, \alpha_i = \min_k \alpha_k\}$, $\delta_i = |x_i - y_i|$, $\vec{\delta} = \{\delta_i\}, i = 1, 2, \dots, d$;

$$\vec{x}^{\vec{\alpha}} := \prod_{i=1}^d x_i^{\alpha_i}, \quad \vec{\delta}^{\pm 1/p} := \left[\prod_{i=1}^d \delta_i \right]^{\pm 1/p},$$

$$G_{\vec{\alpha}}[f](x, y) = \frac{\square[f](x, y)}{|(\vec{x} - \vec{y})^{\vec{\alpha}}|}, \quad \nu(dx, dy) = \frac{d\vec{x}d\vec{y}}{|x - y|},$$

$$\|f\|W(\vec{\alpha}, p) = |G_{\vec{\alpha}}[f](\cdot, \cdot)|_{p, \nu, D^2}.$$

The norm $\|\cdot\|W(\vec{\alpha}, p)$, more precisely, semi-norm is said to be *multidimensional fractional Sobolev's norm* or similar *Aronszajn, Gagliardo or Slobodeckij* norm.

Define also the following function

$$\zeta_{\vec{\alpha}}(p) := \|f\|W(\vec{\alpha}, p), \quad (A, B) := \text{supp} [\zeta_{\vec{\alpha}}(\cdot)]$$

and suppose $1 \leq A < B \leq \infty$.

Denote $A(\vec{\alpha}) = \max(A, p_0)$ and suppose also $A(\vec{\alpha}) < B$;

$$Q_{\alpha, d}(p) = 8^d \cdot 4^{d/p} \cdot \prod_{k=1}^d \left[\frac{\alpha_k + 1/p}{\alpha_k - 1/p} \right].$$

We define a new psi-function $\psi_{\alpha}(p)$ as follows.

$$\psi_{\vec{\alpha}}(p) := \zeta_{\vec{\alpha}}(p) \cdot Q_{\alpha, d}(p).$$

Let $\xi = \xi(x)$ be again random field. We introduce the following *natural* Ψ function: $\theta_{\vec{\alpha}}(p) =$

$$\theta_{\alpha}(p) = Q_{\alpha, d}(p) \cdot \left[\int_D \int_D \mathbf{E} |G_{\vec{\alpha}}[\xi](x, y)|^p \nu(dx, dy) \right]^{1/p}, \quad (5.7)$$

$$\alpha = \vec{\alpha} = \{\alpha_1, \alpha_2, \dots, \alpha_d\}, \quad \alpha_k = \text{const} > 0;$$

and suppose the function $\theta_\alpha(p)$ has non-trivial support such that

$$A = \inf \text{supp } \theta_\alpha(\cdot) \geq 1/\min_k \alpha_k, \quad B = \sup \text{supp } \theta_\alpha \in (A, \infty].$$

Theorem 5.1.

Let $\nu(p) = \nu_\alpha(p)$ be some function from the set $\Psi(A, B)$ such that the function $\gamma(p) = \gamma_\alpha(p) = \nu(p)/\theta_\alpha(p)$ belongs also to the set $\Psi(A, B)$. Then

$$||\Omega(\xi, \delta)||G\nu \leq \delta^\alpha \cdot \phi(G\gamma, 1/\delta). \quad (5.8)$$

Proof. We will use the following result from [31]:

$$|\Omega(\xi, \delta)|_p \leq \delta^{\alpha-1/p} \theta_\alpha(p), \quad p \in (A, B), \quad (5.9)$$

from which follows

$$\frac{|\Omega(\xi, \delta)|_p}{\nu(p) \cdot \delta^\alpha} \leq \frac{(1/\delta)^{1/p}}{\gamma(p)}. \quad (5.10)$$

It remains to take supremum over p ; $p \in (A, B)$ from both the sides of the last inequality (5.10).

Theorem 5.2. Denote

$$\theta_{\alpha,R}(p) = \frac{C_R p}{e \cdot \ln p} \cdot \theta_\alpha(p). \quad (5.11)$$

Let $\nu(p) = \nu_\alpha(p)$ be some function from the set $\Psi(A, B)$ such that the function $\gamma_R(p) = \gamma_{\alpha,R}(p) = \nu(p)/\theta_{\alpha,R}(p)$ belongs also to the set $\Psi(A, B)$. Then

$$||\Omega(S_n, \delta)||G\nu \leq \delta^\alpha \cdot \phi(G\gamma_R, 1/\delta). \quad (5.12)$$

Proof is alike to one in theorem 5.1., in which we substitute the r.f. $S_n(\cdot)$ instead the r.f. $\xi(\cdot)$ and apply the Rosenthal's inequality.

Definition 5.1 of the rectangle Hölder space.

Let $f : D \rightarrow R$ be continuous function and let $\omega = \omega(\delta) = \omega(\vec{\delta})$, $0 \leq \delta_i \leq 1$ be some non - trivial rectangle module of continuity, i.e. non - negative continuous monotonically increasing over each variable δ_i function such that

$$\omega(\delta) = 0 \Leftrightarrow \exists i = 1, 2, \dots, d : \delta_i = 0. \quad (5.13)$$

Define the following rectangle Hölder's norm

$$||f||_{H_r(\omega)} \stackrel{\text{def}}{=} \sup_{x \in D} |f(x)| + \sup_{\delta > 0} \left[\frac{\Omega(f, \delta)}{\omega(\delta)} \right], \quad (5.14)$$

and correspondingly the rectangle Hölder's space $H_r(\omega)$ which consists on all the (continuous) functions $f : D \rightarrow R$ with finite norm $\|f\|_{H_r(\omega)}$.

This space is not separable, therefore we define the (closed) its subspace (separable component) $H_r^o(\omega)$ consisting on all the function from the space $H_r(\omega)$ satisfying the additional condition

$$\lim_{|\delta| \rightarrow 0+} \left[\frac{\Omega(f, \delta)}{\omega(\delta)} \right] = 0,$$

under at the same norm.

It follows immediately from theorem 5.2 the following assertion.

Theorem 5.3. We retain all the notations and conditions of theorem 5.2. Suppose that the module of continuity $\omega_0 = \omega_0(\delta)$ be such that

$$\lim_{|\delta| \rightarrow 0} \left\{ \frac{\omega_0(\delta)}{\delta^\alpha \cdot \phi(G\gamma_R, 1/\delta)} \right\} = \infty. \quad (5.15)$$

Then the r.f. $\xi(x)$ satisfies the CLT in the rectangle Hölder's space $H_r^o(\omega_0)$.

Authors does not know another versions of the CLT in the rectangle Hölder's spaces.

6 Reducing of degree.

Let $X = [0, 1]^m$, $m = 2, 3, \dots$. In the articles [24], [42], [18] is obtained under some additional conditions (condition 2.5 etc.) a multivariate generalization of famous Garsia-Rodemich-Rumsey inequality [15]. Roughly speaking, instead degree "2" in our inequalities stands degree 1 and coefficients dependent on the distance d .

The ultimate (sharp) value of this degree in general case of arbitrary metric space (X, d) is now unknown; see also [2], [19].

We intend to generalize the statement of theorem 3.1 on the case when the Young (Young-Orlicz) function Φ satisfies in addition the condition (2.5) and is twice continuous differentiable.

Some new notations. As in the second section

$$d = d_\Phi = d_\Phi(x_1, x_2) := \|\xi(x_1) - \xi(x_2)\| L(\Phi).$$

Further,

$$K(\Phi) := \sup_{x, y > 0} \left[\frac{\Phi^{-1}(xy)}{\Phi^{-1}(x) + \Phi^{-1}(y)} \right] < \infty; \quad C(\Phi) := \frac{\Phi^{-1}(1)}{54K^2(\Phi)}; \quad (6.1)$$

Let us define also the following important distance function: $\tau(x_1, x_2) =$

$$\tau(x_1, x_2; \Phi) = \tau(x_1, x_2; \Phi, m) = \tau(x_1, x_2; \Phi, m, d) =$$

$$\max \left\{ \int_0^{d(x_1, x_2)} \Phi^{-1} \left[\frac{1}{m(B(r, x_1))} \right] dr, \int_0^{d(x_1, x_2)} \Phi^{-1} \left[\frac{1}{m(B(r, x_2))} \right] dr \right\}. \quad (6.2)$$

If $\forall x_1, x_2 \in X \Rightarrow \tau(x_1, x_2) < \infty$, then the measure $m(\cdot)$ is called *weakly majorizing*; in this case the function $\tau = \tau(x_1, x_2)$ satisfies the triangle inequality and other properties of the distance function [24].

Let the function $\phi(\cdot)$ be natural function for the r.f. $\xi(x)$, $x \in X$:

$$\phi(\lambda) = \log \sup_{x \in X} \mathbf{E} \exp(\lambda \xi(x)),$$

if obviously the family of the centered r.v. $\{\xi(x), x \in X\}$ satisfies the *uniform* Kramer's condition. Denote

$$\bar{\phi}(\lambda) = \sup_{n=1,2,\dots} [n\phi(\lambda/\sqrt{n})], \quad \bar{\Phi}(u) = \exp(\bar{\phi}^*(u)) - 1. \quad (6.3)$$

Theorem 6.1. Let $m(\cdot)$ be some weakly majorizing measure relative the Young function $\bar{\Phi}(\cdot)$. Suppose also $K(\bar{\Phi}) < \infty$. Let also $\theta = \theta(x_1, x_2)$ be arbitrary distance function such that $\tau_{\bar{\Phi}} \ll \theta$.

Then the r.f. $\xi(x)$ satisfies CLT in Hölder space $H^o(\theta)$.

Proof. The inequality

$$\left\| 0.5 C(\Phi) \sup_{\tau(x_1, x_2) > 0} \frac{|\xi(x_1) - \xi(x_2)|}{\tau(x_1, x_2)} \right\| L(\Phi) \leq 1 \quad (6.4)$$

for the r.f. $\xi(x)$ is in fact proved in [24]; see also [37]. As we knew, $\Phi(u) = \exp(\phi^*(u)) - 1$.

We apply the last inequality (6.4) for the random field $S_n(\cdot)$. Let us estimate first of all the moment generating function for the r.v. $S_n(x_1) - S_n(x_2)$.

Recall preliminarily that if $\{\eta_i\}$, $i = 1, 2, \dots$ be a sequence of i.i.d. centered r.v. satisfying the Kramer's condition:

$$\exists \lambda_0 \in (0, \infty], \forall \lambda : |\lambda| < \lambda_0 \Rightarrow \phi(\lambda) := \log \mathbf{E} \exp(\lambda \eta_1) < \infty,$$

then

$$\sup_n \log \mathbf{E} \exp \left(n^{-1/2} \sum_{i=1}^n \eta_i \right) \leq \bar{\phi}(\lambda), \quad |\lambda| < \lambda_0. \quad (6.5)$$

We can rewrite the inequality (6.5) taking into account the relation between the functions ϕ and Φ as follows

$$\sup_n \left\| n^{-1/2} \sum_{i=1}^n \eta_i \right\| L(\bar{\Phi}) \leq C_1 \|\eta_1\| L(\Phi). \quad (6.6)$$

Therefore,

$$\sup_n \|S_n(x_1) - S_n(x_2)\| L(\bar{\Phi}) \leq C_1 d_{\Phi}(x_1, x_2), \quad (6.7)$$

and we conclude by means of the estimate (6.4)

$$\sup_n \left\| C(\Phi) \sup_{\tau(x_1, x_2) > 0} \frac{|S_n(x_1) - S_n(x_2)|}{\tau(x_1, x_2)} \right\| L(\overline{\Phi}) \leq 1. \quad (6.8)$$

It remains to repeat the arguments using by the proof of theorem 3.1.

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